Due Sun

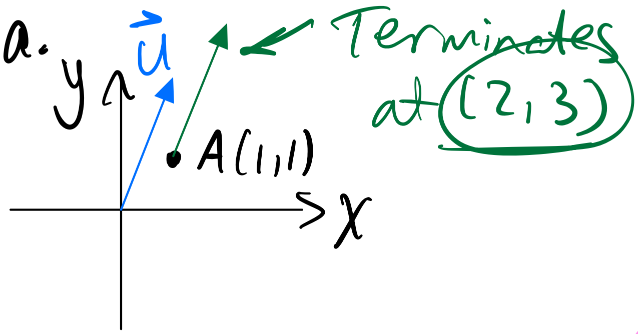
3.1 – Vectors in 2-Space, 3-Space, and n-Space

Definition 1: If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(v_1, v_2, \dots, v_n)$. The set of all ordered n-tuples is called **real n-space**, denoted by R^n.

If a vector $\mathbf{v}$ in 2-space or 3-space is positioned with its initial point at the origin of a rectangular coordinate system, then the coordinates of its terminal point are **components** of $\mathbf{v}$ relative to the coordinate system. The numbers in an n-tuple can be considered coordinates of a point or components of a vector.

5. a. Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose initial point is $A(1, 1)$.
b. Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$.

a.



$\vec{u} = (2-1, 3-1)$

terminal − initial

b. initial: $A(x_1, y_1, z_1)$

$\vec{u} = \overrightarrow{AB}$

$(1, 1, 3) = (-1 - x_1, -1 - y_1, 2 - z_1)$

$1 = -1 - x_1$ $\quad 1 = -1 - y_1$ $\quad 3 = 2 - z_1$

$x_1 = -2$ $\quad y_1 = -2$ $\quad z_1 = -1$

$A(x_1, y_1, z_1) = (-2, -2, -1)$

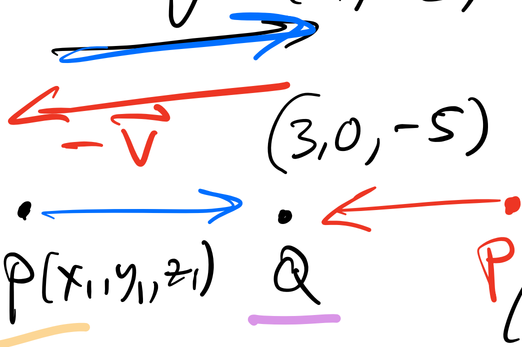
7. Find the initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3, 0, -5)$ and such that

a. $\mathbf{u}$ has the same direction as $\mathbf{v} = (4, -2, -1)$.

b. $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (4, -2, -1)$.

$\vec{v} = (4, -2, -1)$

a.



$\vec{v} = (3-x_1, 0-y_1, -5-z_1)$

$(4,-2,-1) = (3-x_1, -y_1, -5-z_1)$

$4 = 3-x_1 \quad -2 = -y_1, \quad -1 = -5-z_1$

$P(-1, 2, -4)$

b) $-\vec{v} = (-4, 2, 1) = (3-x_1, -y_1, -5-z_1)$

$P(7, -2, -6)$

Definition 3: If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$
  [this is **vector addition**]
- $k\mathbf{v} = (kv_1, kv_2, \dots, kv_n)$
  [this is **scalar multiplication**]
- $-\mathbf{v} = (-v_1, -v_2, \dots, -v_n)$
- $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \dots, w_n - v_n)$

12. Let $\mathbf{u} = (1, 2, -3, 5, 0)$, $\mathbf{v} = (0, 4, -1, 1, 2)$, and $\mathbf{w} = (7, 1, -4, -2, 3)$. Find the components of

a. $\mathbf{v} + \mathbf{w}$

b. $3(2\mathbf{u} - \mathbf{v})$

c. $(3\mathbf{u} - \mathbf{v}) - (2\mathbf{u} + 4\mathbf{w})$

d. $\frac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v}$

a) $\vec{v} + \vec{w} = (0+7,\ 4+1,\ -1+(-4),\ 1+(-2),\ 2+3)$

$= (7, 5, -5, -1, 5)$

and so on

The **zero vector**, $\mathbf{0} = (0, 0, \dots, 0)$, has length zero.

$R^2: \vec{0} = (0, 0)$ $R^3: \vec{0} = (0, 0, 0)$

Theorem 3.1.1 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k and m are scalars, then:

a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = \mathbf{u} + \mathbf{v} + \mathbf{w}$

c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

g) $k(m\mathbf{u}) = (km)\mathbf{u}$

h) $1\mathbf{u} = \mathbf{u}$

Pf (f): Let $\vec{u} = (u_1, u_2, \dots, u_n) \in R^n$ and k, m be scalars.

$$(k+m)\vec{u} = ((k+m)u_1, (k+m)u_2, \dots, (k+m)u_n)$$
(def of scalar mult.)

$$= (ku_1 + mu_1, ku_2 + mu_2, \dots, ku_n + mu_n)$$
(dist prop. for mult of real #s)

$$= (ku_1, ku_2, \dots, ku_n) + (mu_1, mu_2, \dots, mu_n)$$
(def of vector addition)

$$= k(u_1, u_2, \dots, u_n) + m(u_1, u_2, \dots, u_n)$$
(def of scalar mult)

$$= k\vec{u} + m\vec{u} \quad ✓$$

Theorem 3.1.2 If $\mathbf{v}$ is a vector in R^n and k is a scalar, then:

a) $0\mathbf{v} = \mathbf{0}$

b) $k\mathbf{0} = \mathbf{0}$

c) $(-1)\mathbf{v} = -\mathbf{v}$

Definition 4: if $\boldsymbol{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in R^n if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$, where each k_i is a scalar. The scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, we have $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is a scalar multiple of the vector.

$3\vec{u} - \vec{v} = (3,-3,9,15) + (-2,-1,0,3) = (1,-4,9,18)$ ✓

17. Let $\mathbf{u} = (1,-1,3,5)$ and $\mathbf{v} = (2,1,0,-3)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (1,-4,9,18)$.

$$a(1,-1,3,5) + b(2,1,0,-3) = (1,-4,9,18)$$

Rewrite: $\begin{bmatrix} 1 \\ -1 \\ 3 \\ 5 \end{bmatrix} a + \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} b = \begin{bmatrix} 1 \\ -4 \\ 9 \\ 18 \end{bmatrix}$ (overkill, but this lets us use matrices)

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 0 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 9 \\ 18 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 3 & 0 & 9 \\ 5 & -3 & 18 \end{array}\right]$$

We find $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\rightarrow a = 3$, $\rightarrow b = -1$

Note: A contradiction would indicate that no such a & b exist.